COHERENT SYSTEMS OF GENUS 0 II: EXISTENCE RESULTS FOR $k \geq 3$

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Abstract. In this paper we continue the investigation of coherent systems of type (n, d, k) on the projective line which are stable with respect to some value of a parameter α. We work mainly with $k < n$ and obtain existence results for arbitrary k in certain cases, together with complete results for $k = 3$. Our methods involve the use of the "flips" which occur at critical values of the parameter.

1. Introduction

A *coherent system of type* (n, d, k) on a smooth projective curve C over an algebraically closed field is by definition a pair (E, V) with E a vector bundle of rank n and degree d over C and $V \subset H^0(E)$ a linear subspace of dimension k. For any real number α, the α-*slope* of a coherent system (E, V) of type (n, d, k) is defined by

$$\mu_\alpha(E, V) := \frac{d}{n} + \alpha\frac{k}{n}.$$

A *coherent subsystem* of (E, V) is a coherent system (F, W) such that F is a subbundle of E and $W \subset V \cap H^0(F)$. A coherent system is called α-*stable* (α-*semistable*) if

$$\mu_\alpha(F, W) < \mu_\alpha(E, V) \quad (\mu_\alpha(F, W) \leq \mu_\alpha(E, V))$$

for every proper coherent subsystem (F, W) of (E, V). According to general theory (see, for example, [1]), there exists a moduli space of α-stable coherent systems of type (n, d, k), which we denote by $G(\alpha; n, d, k)$.

In a previous paper [2], we obtained necessary conditions for the existence of α-stable coherent systems of type (n, d, k) on a curve of genus 0. We showed further that these conditions were also sufficient when $k = 1$, but for $k = 2$ a special case $(n, d) = (4, 6)$ had to be excluded. In this paper we show that, when $k < n$, the conditions of

2000 *Mathematics Subject Classification.* Primary: 14H60; Secondary: 14F05, 32L10.

Key words and phrases. Vector bundle, coherent system, moduli space.

Both authors are members of the research group VBAC (Vector Bundles on Algebraic Curves). They were supported by the Forschungsschwerpunkt "Globale Methoden in der komplexen Analysis" of the DFG. The second author would like to thank the Mathematisches Institut der Universität Erlangen-Nürnberg for its hospitality.

[2] remain sufficient for the existence of α-stable coherent systems for small positive values of α (we write this as $\alpha = 0^+$). For arbitrary α, this is no longer true, but we can prove that, for each fixed value of k, there are only finitely many pairs (n, d) for which exceptional behaviour occurs. When $k = 3$, there are indeed exceptional cases where the range of α for which α-stable coherent systems exist is strictly smaller than the range shown to be necessary in [2]. We analyse these cases and obtain necessary and sufficient conditions for existence. We give also an example with $k = 4$ to show that, in higher ranks, further complications arise.

We have two principal methods. The first is a development of an argument used in [2], whereby the existence problem for small positive values of α is reduced to a problem in projective geometry, which we solve completely. The second method is completely different from those of [2], depending on an analysis of the "flips" introduced in [1]. The advantage of this approach is that it makes it possible to translate results from one value of α to another. It also allows us to construct α-stable coherent systems starting from α-stable (or even α-semistable) coherent systems of lower rank.

We now outline the content of the paper including statements of the main results (for notations, see section 2 or [2]). We begin in section 2 by describing the general set-up and establishing notation. This is followed in section 3 by a detailed strategy for the analysis of flips. The case where d is a multiple of n is considered in section 4, where we prove

Theorem 4.5. *Suppose $0 < k < n$. Then there exists a 0^+-stable coherent system of type (n, na, k) if and only if*

$$ka \geq n - k + \frac{k^2 - 1}{n}.$$

In section 5, we introduce the concept of an allowable critical data set and carry out our first computations of the numbers C_{12} and C_{21} associated with the corresponding flips. We prove in particular the following general result:

Theorem 5.8. *Let k be a fixed positive integer. Then there are only finitely many allowable critical data sets with $n > k$ for which $C_{12} \leq 0$ or $C_{21} \leq 0$.*

For $k < n$ we write as in [2]

$$d = na - t \quad \text{and} \quad ka = l(n - k) + t + m$$

with $0 \leq t < n$ and $0 \leq m < n - k$. We then obtain as a consequence of Theorem 5.8:

Corollary 5.9. *Let k be a fixed positive integer. Then, for all but finitely many pairs (n, d) with $n > k$, one of the following two possibilities holds:*

- $G(\alpha; n, d, k) = \emptyset$ *for all* α*;*
- $G(\alpha; n, d, k) \neq \emptyset$ *for all* α *such that*

$$\frac{t}{k} < \alpha < \frac{ln + t}{k}.$$

The inequalities for α in this corollary are precisely the necessary conditions of [2, Propositions 4.1 and 4.2]. This result therefore justifies our assertion that, for each value of k, there are only finitely many pairs (n, d) which exhibit exceptional behaviour. Improved results can be obtained when $t = 0$ (i. e. when d is a multiple of n), $t = 1$ or $t \geq k - 1$ (Corollaries 5.10 and 5.13).

In section 6, we reprove the results of [2] for $k = 2$ using our new techniques. The following three sections contain our results for $k = 3$. The main result is

Theorem 8.4. *Suppose $n \geq 4$. Then $G(\alpha; n, d, 3) \neq \emptyset$ for some $\alpha > 0$ if and only if $l \geq 1$, $d \geq \frac{1}{3}n(n-3) + \frac{8}{3}$ and $(n, d) \neq (6, 9)$. Moreover, when these conditions hold, $G(\alpha; n, d, 3) \neq \emptyset$ if and only if*

$$\frac{t}{3} < \alpha < \frac{d}{n-3} - \frac{mn}{3(n-3)},$$

except for the following pairs (n, d), where the range of α is as stated :

$$\begin{array}{llll} for & (4,7): \ \frac{3}{5} < \alpha < 7; & for & (5,9): \ \frac{3}{4} < \alpha < \frac{11}{3}; \\ for & (6,11): \ 1 < \alpha < \frac{7}{3}; & for & (7,13): \ \frac{3}{2} < \alpha < \frac{8}{3}. \end{array}$$

For completeness, we also discuss the case $n \leq 3$, obtaining

Theorem 9.2.

(i): *$G(\alpha; 1, d, 3) \neq \emptyset$ if and only if $d \geq 2$ and $\alpha > 0$.*

(ii): *$G(\alpha; 2, d, 3) \neq \emptyset$ for some α if and only if $d \geq 2$. Moreover, if $d \geq 2$, $G(\alpha; 2, d, 3) \neq \emptyset$ for all $\alpha > \frac{t}{3}$ except in the case $d = 3$, when $G(\alpha; 2, 3, 3) \neq \emptyset$ if and only if $\alpha > 1$.*

(iii): *$G(\alpha; 3, d, 3) \neq \emptyset$ for some α if and only if $d \geq 4$. Moreover, if $d \geq 4$, $G(\alpha; 3, d, 3) \neq \emptyset$ for all $\alpha > \frac{t}{3}$ except in the case $d = 5$, when $G(\alpha; 3, 5, 3) \neq \emptyset$ if and only if $\alpha > \frac{2}{3}$.*

Finally, in section 10, we give an example of an allowable critical data set with $k = 4$ where $C_{12} = 0$ (this is the smallest value of k for which such a critical data set exists).

We work throughout on the projective line $\mathbb{P}^1$ defined over an algebraically closed field $\mathbb{K}$.

2. The set up

Let $G(\alpha; n, d, k)$ denote the moduli space of α-stable coherent systems on $\mathbb{P}^1$ of type (n, d, k). We recall [2, Theorem 3.2] that, when it

is non-empty, $G(\alpha; n, d, k)$ is always irreducible of dimension

$$\beta(n, d, k) := -n^2 + 1 - k(k - d - n). \tag{1}$$

In particular, if $G(\alpha; n, d, k) \neq \emptyset$, then $\beta(n, d, k) \geq 0$.

In accordance with [1, section 6], we consider exact sequences

$$0 \to (E_1, V_1) \to (E, V) \to (E_2, V_2) \to 0 \tag{2}$$

and

$$0 \to (E_2, V_2) \to (E', V') \to (E_1, V_1) \to 0 \tag{3}$$

with (E, V) and (E', V') of type (n, d, k) and (E_i, V_i) of type (n_i, d_i, k_i) for $i = 1, 2$. We suppose also that

$$\frac{d_2}{n_2} > \frac{d_1}{n_1} \quad \text{and} \quad \frac{k_1}{n_1} > \frac{k_2}{n_2} \tag{4}$$

and define

$$\alpha_c = \frac{d_2 n - d n_2}{n_2 k - n k_2} = \frac{d_2 n_1 - d_1 n_2}{n_2 k_1 - n_1 k_2}. \tag{5}$$

We write α_c^- for a value of α slightly smaller than α_c and α_c^+ for a value of α slightly larger than α_c. We suppose always that (E_1, V_1) and (E_2, V_2) are both α_c-semistable. Note that

$$\mu_{\alpha_c}(E_1, V_1) = \mu_{\alpha_c}(E_2, V_2) = \frac{d}{n} + \alpha_c \frac{k}{n},$$

so (E, V) is strictly α_c-semistable.

Given this set-up, we shall refer to α_c as a *critical value* and to

$$A_c := (\alpha_c, n_1, d_1, k_1, n_2, d_2, k_2)$$

as a *critical data set*. Note that, given (n, d, k), a critical data set is determined by giving values to (n_2, d_2, k_2) but not necessarily simply by the critical value α_c. Essentially a critical value α_c is a value of α at which the α-stability condition for a coherent system (E, V) can change as α passes through the value α_c, while the corresponding critical data sets describe the way in which this change takes place. For convenience we write $G(\alpha_c^-) := G(\alpha_c^-; n, d, k)$ and $G_{\alpha_c}^-$ for the "flip locus" in $G(\alpha_c^-)$, that is the closed subvariety consisting of those coherent systems which are α_c^--stable but not α_c^+-stable. Similarly we define $G(\alpha_c^+)$ and $G_{\alpha_c}^+$ with $+$ and $-$ interchanged.

As in [1] (and putting $g = 0$), we define for any critical data set

$$C_{12} = -n_1 n_2 - d_2 n_1 + d_1 n_2 + k_1(d_2 + n_2 - k_2) \tag{6}$$

and

$$C_{21} = -n_1 n_2 + d_2 n_1 - d_1 n_2 + k_2(d_1 + n_1 - k_1). \tag{7}$$

We shall explain the significance of C_{12} and C_{21} more precisely in section 3.

We shall be mainly concerned with the case $0 < k < n$. We then write as in the introduction

$$d = na - t \quad \text{and} \quad ka = l(n-k) + t + m$$

with $0 \le t < n$ and $0 \le m < n - k$. Note that, by [2, Remark 4.3], $l > 0$ is a necessary condition for $G(\alpha; n, d, k)$ to be non-empty. From (4), we have $d_2 > n_2\frac{d}{n} = n_2 a - \frac{n_2}{n} t$ and we write

$$d_2 = n_2 a + e \tag{8}$$

with an integer $e > -\frac{n_2}{n} t$. Using (8), we can rewrite (5) as

$$\alpha_c = \frac{ne + n_2 t}{n_2 k - n k_2}. \tag{9}$$

According to [2, Propositions 4.1 and 4.2], we can suppose

$$\frac{t}{k} < \alpha_c < \frac{d}{n-k} - \frac{mn}{k(n-k)} = \frac{ln + t}{k}, \tag{10}$$

which in terms of e means

$$-\frac{k_2}{k} t < e < -\frac{k_2}{k} t + l\left(n_2 - \frac{k_2}{k} n\right). \tag{11}$$

Note that $-\frac{k_2}{k} t \ge -\frac{n_2}{n} t$ (with equality if and only if $t = 0$), so this inequality is stronger than $e > -\frac{n_2}{n} t$. Now write

$$e = -\frac{k_2}{k} t + l\left(n_2 - \frac{k_2}{k} n\right) - \frac{f}{k} \tag{12}$$

with an integer $f \ge 1$. In particular

$$f \equiv -k_2(t + ln) \equiv k_2 m \bmod k. \tag{13}$$

3. The strategy

In this section we explain our strategy for analysing flips. The basic idea (introduced in [1]) is to estimate the numbers C_{12} and C_{21} (see (6) and (7)) for any critical data set and use this information to determine how the α-stability of a coherent system can change as α passes through a critical value. We can also use this approach to construct α-stable coherent systems for values of α close to this critical value.

Let $A_c := (\alpha_c, n_1, d_1, k_1, n_2, d_2, k_2)$ be a critical data set. We consider the exact sequences of the forms (2), (3), where as usual we suppose that (E_1, V_1) and (E_2, V_2) are both α_c-semistable. Our main object in this section is to show that, in some important cases, the inequalities for the codimensions of the flip loci given in [1, equations (17) and (18)] can be replaced by equalities. We begin with a version of [2, Lemma 3.1] for α-semistability.

Lemma 3.1. *Suppose $k > 0$ and (E, V) is α-semistable for some $\alpha > 0$. Then*

$$E \simeq \bigoplus_{i=1}^{n} \mathcal{O}(a_i)$$

with all $a_i \geq 0$.

Proof. We can write $E = F \oplus G$, where every direct factor of F has negative degree and every direct factor of G has non-negative degree. Since $H^0(F) = 0$, it follows that

$$(E, V) = (F, 0) \oplus (G, V).$$

If $F \neq 0$, then $\mu_\alpha(F, 0) < 0$ for all α, while $\mu_\alpha(G, V) > 0$ for $\alpha > 0$. This contradicts the α-semistability of (E, V). □

Corollary 3.2. *Suppose that (E_1, V_1) and (E_2, V_2) are both α-semistable. Then*

$$\mathrm{Ext}^2((E_1, V_1), (E_2, V_2)) = \mathrm{Ext}^2((E_2, V_2), (E_1, V_1)) = 0. \tag{14}$$

Proof. This follows from the lemma, together with the formula for Ext^2 given in [1, equation (11)] and the fact that the canonical bundle has negative degree. □

Corollary 3.3. *Let α_c be a critical value. Suppose that, for every critical data set A_c with critical value α_c, we have $C_{12} > 0$ and $C_{21} > 0$. Then $G(\alpha_c^+)$ is birational to $G(\alpha_c^-)$. In fact, if $C_{12} > 0$ for every A_c and $G(\alpha_c^-) \neq \emptyset$, then $G_{\alpha_c}^-$ has positive codimension in $G(\alpha_c^-)$. Similarly, if $C_{21} > 0$ for every A_c and $G(\alpha_c^+) \neq \emptyset$, then $G_{\alpha_c}^+$ has positive codimension in $G(\alpha_c^+)$.*

Proof. Since all non-empty moduli spaces have the expected dimensions (given by (1)), it follows from Corollary 3.2 and [1, equations (17) and (18)] that the flip loci have positive codimensions. The result follows. □

The key fact about the numbers C_{12} and C_{21} is that they play two rôles, as estimates for codimensions of flip loci and for dimensions of spaces of extensions. In fact, if we assume in addition to (14) that

$$\mathrm{Hom}((E_1, V_1), (E_2, V_2)) = \mathrm{Hom}((E_2, V_2), (E_1, V_1)) = 0, \tag{15}$$

then we deduce at once from [1, equation (8)] that

$$C_{12} = \dim \mathrm{Ext}^1((E_1, V_1), (E_2, V_2)) \tag{16}$$

and

$$C_{21} = \dim \mathrm{Ext}^1((E_2, V_2), (E_1, V_1)). \tag{17}$$

In particular, if (15) holds, we always have

$$C_{12} \geq 0, \quad C_{21} \geq 0.$$

Lemma 3.4. *Suppose that, for some critical data set A_c, there exist α_c-stable coherent systems (E_1, V_1) and (E_2, V_2), and that A_c is the only critical data set for the critical value α_c . Then*

(a) *if $C_{21} > 0$, the flip locus $G^-_{\alpha_c}$ is irreducible and has codimension C_{12} in $G(\alpha_c^-)$;*
(b) *if $C_{12} > 0$, the flip locus $G^+_{\alpha_c}$ is irreducible and has codimension C_{21} in $G(\alpha_c^+)$.*

Proof. (a): Consider first the non-trivial extensions (2) with (E_1, V_1) and (E_2, V_2) both α_c-stable. It is easy to see that (E, V) has (2) as its unique Jordan-Hölder filtration at α_c. Since $\frac{k_1}{n_1} > \frac{k_2}{n_2}$, it follows also that (E, V) is α_c^--stable. Since (E_1, V_1) and (E_2, V_2) are non-isomorphic and α_c-stable of the same α_c-slope, (15) holds and therefore also (17). These extensions therefore define a non-empty open subset U of $G^-_{\alpha_c}$ of dimension

$$\dim U = \dim G(\alpha_c; n_1, d_1, k_1) + \dim G(\alpha_c; n_2, d_2, k_2) + C_{21} - 1.$$

It follows from [1, Corollary 3.7] that U has codimension C_{12} in $G(\alpha_c^-)$.

It remains to show that $G^-_{\alpha_c}$ is irreducible. For this, note that, by [1, Lemma 6.5(ii)], all elements of $G^-_{\alpha_c}$ come from extensions (2) with (E_1, V_1) and (E_2, V_2) α_c^--stable. Since $\frac{k_1}{n_1} > \frac{k_2}{n_2}$, $\mu_{\alpha_c^-}(E_1, V_1) < \mu_{\alpha_c^-}(E_2, V_2)$. Hence $\mathrm{Hom}((E_2, V_2), (E_1, V_1)) = 0$, which implies (17). The irreducibility of $G^-_{\alpha_c}$ now follows from the irreducibility of the moduli spaces $G(\alpha_c^-; n_1, d_1, k_1)$ and $G(\alpha_c^-; n_2, d_2, k_2)$.

The proof of (b) is similar. □

Corollary 3.5. *Suppose that the hypotheses of the lemma hold. Then one of the following situations occurs:*

- *$C_{12} > 0$ and $C_{21} > 0$: $G(\alpha_c^-)$ and $G(\alpha_c^+)$ are both non-empty and birational to each other;*
- *$C_{12} = C_{21} = 0$: the flip loci are empty and $G(\alpha_c^-) = G(\alpha_c^+)$;*
- *$C_{21} = 0$, $C_{12} > 0$: $G(\alpha_c^-) = \emptyset$, $G(\alpha_c^+) = G^+_{\alpha_c} \neq \emptyset$;*
- *$C_{12} = 0$, $C_{21} > 0$: $G(\alpha_c^+) = \emptyset$, $G(\alpha_c^-) = G^-_{\alpha_c} \neq \emptyset$.*

Proof. For the first part, the non-emptiness follows from the lemma and the birationality is a special case of Corollary 3.3. If $C_{12} = C_{21} = 0$, then (16) and (17) imply that the flip loci are empty; this proves the second part. If $C_{21} = 0$ and $C_{12} > 0$, the lemma implies that $G(\alpha_c^+) = G^+_{\alpha_c} \neq \emptyset$. It now follows from [2, Corollary 3.4] that $G(\alpha_c^-) = \emptyset$. The last part is proved similarly. □

Remark 3.6. In a calculation it may happen that C_{12} or C_{21} comes out to be negative. In this case either (E_1, V_1) or (E_2, V_2) fails to exist and the flip loci are empty.

Remark 3.7. Suppose there is more than one critical data set A_c for a critical value α_c, such that, for each A_c, there exist α_c-stable

coherent systems (E_1, V_1) and (E_2, V_2). We then ignore all A_c for which $C_{12} = C_{21} = 0$ and replace the remaining C_{12} and C_{21} by their minimum values taken over the various A_c. The conclusions of Corollary 3.5 then hold except possibly when both C_{12} and C_{21} have minimum value zero (necessarily for different A_c). It then follows from the proof of Lemma 3.4 that both $G(\alpha_c^-)$ and $G(\alpha_c^+)$ are non-empty, but $G(\alpha_c)$ is empty. This contradicts [2, Corollary 3.4], so this situation can never arise.

Remark 3.8. The conclusions of Remark 3.7 still hold if there are additional critical data sets with critical value α_c, provided these all have $C_{12} > 0$ and $C_{21} > 0$,

On occasion, we shall need to use extensions in which (E_1, V_1) and (E_2, V_2) are not α_c-stable. In this paper, it will be sufficient to consider extensions

$$0 \to (\mathcal{O}(b)^r, 0) \to (E, V) \xrightarrow{p} (E_1, W) \to 0, \tag{18}$$

for certain (E_1, W) with $\mu_{\alpha_c}(E_1, W) = b$ (note that $\mu_\alpha(\mathcal{O}(b)^r, 0) = b$ for all α).

Lemma 3.9. *Suppose that, in* (18)*, either* (E_1, W) *is* α_c*-stable and*

$$\dim \operatorname{Ext}^1((E_1, W), (\mathcal{O}(b), 0)) \geq r \tag{19}$$

or

$$(E_1, W) = \bigoplus_{i=1}^{n-r} (\mathcal{O}(b'), W_i),$$

where the W_i *are distinct subspaces of dimension* 1 *of* $H^0(\mathcal{O}(b'))$ *and*

$$(n - r)b' > r, \quad n - r \geq 2. \tag{20}$$

Then, for the general extension (18)*,* (E, V) *is* α_c^+*-stable.*

Proof. Suppose (F, U) is a subsystem of (E, V) which contradicts α_c^+-stability. Then (F, U) also contradicts α_c-stability. Since (E, V) is α_c-semistable, (F, U) is also α_c-semistable with the same α_c-slope.

In the first case, this implies that the image $p(F, U)$ is either 0 or equal to (E_1, W). If $p(F, U) = 0$, then $(F, U) \subset (\mathcal{O}(b)^r, 0)$ and does not contradict α_c^+-stability of (E, V). So $p(F, U) = (E_1, W)$ and we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & (\mathcal{O}(b)^s, 0) \arrow[r] \arrow[d, hook] & (F, U) \arrow[r] \arrow[d, hook] & (E_1, W) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & (\mathcal{O}(b)^r, 0) \arrow[r] & (E, V) \arrow[r] & (E_1, W) \arrow[r] & 0
\end{tikzcd}$$

with $s < r$. The extensions (18) are classified by r-tuples $(e_1, \ldots, e_r)$ with $e_i \in \operatorname{Ext}^1((E_1, W), (\mathcal{O}(b), 0))$. By (19), the general extension (18) has $e_1, \ldots, e_r$ linearly independent. Thus the diagram above is impossible.

In the second case, note first that

$$\mathrm{Hom}\left((\mathcal{O}(b'),W_i),(\mathcal{O}(b),0)\right)=0.$$

Hence, by (16) and (6),

$$\dim\ \mathrm{Ext}^1((\mathcal{O}(b'),W_i),(\mathcal{O}(b),0)) = -1-b+b'+(b+1)=b'. \tag{21}$$

If $p(F,U)$ is either 0 or (E_1,W), we argue as in the first case. Otherwise note that, since $(\mathcal{O}(b'),W_i)\not\cong(\mathcal{O}(b'),W_j)$ for $i\neq j$, there are only finitely many possible choices for $p(F,U)$ and we can suppose without loss of generality that

$$p(F,U)=\bigoplus_{i=1}^{j}(\mathcal{O}(b'),W_i)$$

for some j with $1\le j\le n-r-1$. If p maps (F,U) isomorphically to $\bigoplus_{i=1}^{j}(\mathcal{O}(b'),W_i)$, then the extension (18) restricted to $\bigoplus_{i=1}^{j}(\mathcal{O}(b'),W_i)$ splits. But in general this does not happen, since by (21)

$$\dim\ \mathrm{Ext}^1\left(\bigoplus_{i=1}^{j}(\mathcal{O}(b'),W_i),(\mathcal{O}(b),0)\right)=jb'>0.$$

We can therefore suppose that the kernel of $(F,U)\to\bigoplus_{i=1}^{j}(\mathcal{O}(b'),W_i)$ has the form $(\mathcal{O}(b)^s,0)$ with $1\le s\le r$. But then (F,U) contradicts α_c^+-stability of (E,V) if and only if

$$\begin{aligned}\mu_{\alpha_c^+}(F,U) &= \frac{bs+b'j}{s+j}+\alpha_c^+\frac{j}{s+j}\\ &\ge \mu_{\alpha_c^+}(E,V)=\frac{br+b'(n-r)}{n}+\alpha_c^+\frac{n-r}{n}\end{aligned}$$

This is equivalent to

$$(bs+b'j)n-(br+b'(n-r))(s+j)\ge\alpha_c^+((n-r)(s+j)-jn),$$

which in turn is equivalent to

$$(b-b')(s(n-r)-jr)\ge\alpha_c^+(s(n-r)-jr).$$

Since $b-b'=\alpha_c$, this reduces to

$$jr\ge s(n-r). \tag{22}$$

Now consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & (\mathcal{O}(b)^s,0) \arrow[r] \arrow[d, hook] & (F,U) \arrow[r] \arrow[d, hook] & \bigoplus_{i=1}^{j}(\mathcal{O}(b'),W_i) \arrow[r] \arrow[d, equal] & 0\\
0 \arrow[r] & (\mathcal{O}(b)^r,0) \arrow[r] \arrow[d, equal] & (F',U') \arrow[r] \arrow[d, hook] & \bigoplus_{i=1}^{j}(\mathcal{O}(b'),W_i) \arrow[r] \arrow[d, hook] & 0\\
0 \arrow[r] & (\mathcal{O}(b)^r,0) \arrow[r] & (E,V) \arrow[r] & \bigoplus_{i=1}^{n-r}(\mathcal{O}(b'),W_i) \arrow[r] & 0
\end{tikzcd}$$

where the lower half is the pull-back diagram which always exists, and the upper half is a push-out diagram the existence of which we have to analyse. The extensions of the middle row are classified by r-tuples $(e_1, ..., e_r)$ with $e_l \in \operatorname{Ext}^1\left(\bigoplus_{i=1}^{j}(\mathcal{O}(b'), W_i), (\mathcal{O}(b), 0)\right)$, which, by (21), is of dimension jb'. Hence, for a general extension (18), such a diagram cannot exist unless $jb' \leq s$. Combining this with (22), we obtain

$$jb'(n-r) \leq s(n-r) \leq jr,$$

which contradicts the hypothesis $(n-r)b' > r$. This completes the proof. □

Remark 3.10. The hypotheses (19) and (20) in the statement of the lemma are sharp. In the first case, if (19) fails, (E, V) has a direct factor of the form $(\mathcal{O}(b), 0)$ and is not even α_c^+-semistable. In the second case, if $(n-r)b' \leq r$, we can take $s = jb'$ to contradict α_c^+-stability. In fact, if $(n-r)b' = r$, the general (E, V) is strictly α_c^+-semistable.

4. The case $t = 0$

In this section we assume $0 < k < n$ and consider the existence problem for 0^+-stable coherent systems of type (n, d, k). Note that, if (E, V) is such a coherent system, the bundle E is semistable, so $E \simeq \mathcal{O}(a)^n$ and $t = 0$. We therefore suppose that $E = \mathcal{O}(a)^n$ and assume also that the homomorphism $\beta : V \otimes \mathcal{O} \to \mathcal{O}(a)^n$ is injective. For $1 \leq q \leq k$, we then define

$$\delta_q(n, a, \beta) = \begin{cases} \text{minimal rank of a direct factor of } \mathcal{O}(a)^n \\ \text{containing the image of some } \mathcal{O}^q \subset V \otimes \mathcal{O} \text{ under} \\ \text{the composed map } \mathcal{O}^q \hookrightarrow V \otimes \mathcal{O} \xrightarrow{\beta} \mathcal{O}(a)^n. \end{cases}$$

Lemma 4.1. *$(\mathcal{O}(a)^n, V)$ is 0^+-stable if and only if $\delta_k(n, a, \beta) = n$ and*

$$\delta_q(n, a, \beta) > \frac{qn}{k} \quad \text{for} \quad 1 \leq q \leq k-1.$$

Proof. Suppose (F, W) is a coherent subsystem of $(\mathcal{O}(a)^n, V)$ with $\dim W = q$. Then (F, W) contradicts the 0^+-stability of $(\mathcal{O}(a)^n, V)$ if and only if $F \simeq \mathcal{O}(a)^r$ where either $q = k$ and $r < n$ or $1 \leq q < k$ and $\frac{q}{r} \geq \frac{k}{n}$, i. e. $\frac{qn}{k} \geq r$. The result follows from the definition of δ_q. □

We now convert this condition into a statement in projective geometry. For this, let q, k and n denote positive integers with $q \leq k < n$ and consider the Segre embedding

$$\mathbb{P}^{k-1} \times \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^{kn-1}.$$

For any integer a with $0 \leq a \leq kn-2$, let $R(n, a, k, q)$ denote the maximum number r such that any linear subspace $W \subset \mathbb{P}^{kn-1}$ of codimension $a+1$ contains some subspace $\mathbb{P}^{q-1} \times \mathbb{P}^{r-1} \subset \mathbb{P}^{k-1} \times \mathbb{P}^{n-1} \subset \mathbb{P}^{kn-1}$. If $a \geq kn-1$, we define $R(n, a, k, q) = 0$. Note that the condition on

W is equivalent to saying that W contains the subspace $\mathbb{P}^{qr-1} \subset \mathbb{P}^{kn-1}$ which is spanned by the image of $\mathbb{P}^{q-1} \times \mathbb{P}^{r-1}$ in $\mathbb{P}^{kn-1}$.

Lemma 4.2. *For a general choice of $V \subset H^0(\mathcal{O}(a)^n)$,*

$$\delta_q(n, a, \beta) = n - R(n, a, k, q).$$

Proof. The map β is given by a matrix of the form

$$M = (f_{ij})_{1 \leq i \leq n, 1 \leq j \leq k}$$

where the f_{ij} are binary forms of degree a. The composition $\mathcal{O}^q \to \mathcal{O}(a)^n$ is given by a matrix MN_q of rank q with

$$N_q = (b_{jp})_{1 \leq j \leq k, 1 \leq p \leq q}$$

where the b_{jp} are constants and rk $N_q = q$. By definition of $\delta_q(n, a, \beta)$ we have

$$n - \delta_q(n, a, \beta) = \max_{A \in GL(n, \mathbb{C}),\ rk\ N_q = q} \{\text{number of zero rows in } AMN_q\}.$$

But this equals the maximum number of linearly independent vectors $(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ such that

$$(\lambda_1, \cdots, \lambda_n) MN_q = 0,$$

the maximum to be taken over all $k \times q$-matrices N_q of rank q.

Now let W denote the projectivisation of the kernel of the linear map $\mathbb{C}^{kn} \longrightarrow H^0(\mathcal{O}(a))$ given by

$$(\mu_{11}, \dots, \mu_{1k}, \dots, \mu_{n1}, \dots, \mu_{nk}) \mapsto \sum f_{ij} \mu_{ij}.$$

Note that, if $a \leq kn - 2$, then, for a general choice of the f_{ij}, W has codimension $a + 1$ in $\mathbb{P}^{kn-1}$. The result follows easily from the definitions of δ_q and R. $\square$

The next step is to estimate $R(n, a, k, q)$.

Lemma 4.3.

$$R(n, a, k, q) \leq \left\lfloor \frac{1}{2} \left(n - q(a+1) + \sqrt{(n - q(a+1))^2 + 4q(k-q)} \right) \right\rfloor.$$

Proof. For $a \geq kn-1$, this is obvious since $R(n, a, k, q) = 0$. Otherwise, let $Gr := Gr(kn - a - 1, kn)$ denote the Grassmannian of subspaces of codimension $a+1$ in $\mathbb{P}^{kn-1}$. For a fixed linear subspace $\mathbb{P}^{qr-1} \subset \mathbb{P}^{kn-1}$, let Σ denote the closed subspace of Gr consisting of all $W \in Gr$ with $\mathbb{P}^{qr-1} \subset W$. Finally write $\Psi := Gr(q, k) \times Gr(r, n)$.

We can clearly ignore the values of r for which $\Sigma = \emptyset$, or equivalently $a \geq kn - qr$. Otherwise, a necessary condition for a general subspace $W \subset \mathbb{P}^{kn-1}$ of codimension $a+1$ to contain some subspace $\mathbb{P}^{q-1} \times \mathbb{P}^{r-1} \subset \mathbb{P}^{k-1} \times \mathbb{P}^{n-1} \subset \mathbb{P}^{kn-1}$ is

$$\dim \Sigma + \dim \Psi \geq \dim Gr.$$

Since

$$\begin{aligned}\dim Gr &= (kn-a-1)(a+1)\\ \dim \Sigma &= (kn-a-1-qr)(a+1)\\ \dim \Psi &= q(k-q)+r(n-r),\end{aligned}$$

this means

$$(kn-a-1-qr)(a+1)+q(k-q)+r(n-r) \geq (kn-a-1)(a+1),$$

which is equivalent to

$$r^2+(q(a+1)-n)r-q(k-q) \leq 0.$$

This quadratic equation in r always has two real solutions. Solving this equation gives the assertion. □

Corollary 4.4. *$R(n,a,k,k)=0$ if $ka \geq n-k$.*

Proof. The hypothesis states that $n-k(a+1) \leq 0$. The assertion then follows immediately from the lemma. □

Theorem 4.5. *Suppose $0 < k < n$. Then there exists a 0^+-stable coherent system of type (n,na,k) if and only if*

$$ka \geq n-k+\frac{k^2-1}{n}. \tag{23}$$

Proof. Note first that (23) is equivalent to the Brill-Noether inequality $\beta(n,na,k) \geq 0$ (see (1)). The inequality (23) is therefore a necessary condition for the existence of α-stable coherent systems of type (n,na,k).

Conversely, suppose (23) holds. In view of Lemmas 4.1 and 4.2 and Corollary 4.4, it is sufficient to show that, for $1 \leq q \leq k-1$,

$$R(n,a,k,q) < n-\frac{qn}{k}. \tag{24}$$

We prove first

Lemma 4.6. *Suppose*

$$ka > n-k+\frac{k^2}{n}. \tag{25}$$

Then (24) holds for $1 \leq q \leq k-1$.

Proof. By Lemma 4.3, it is sufficient to prove that

$$n-q(a+1)+\sqrt{(n-q(a+1))^2+4q(k-q)} < 2n-\frac{2qn}{k},$$

i. e.

$$\sqrt{(n-q(a+1))^2+4q(k-q)} < n+q(a+1)-\frac{2qn}{k}. \tag{26}$$

We show first that the right-hand side of (26) is positive. In fact

$$n+q(a+1)-\frac{2qn}{k} > 0 \Leftrightarrow k(a+1) > 2n-\frac{nk}{q}.$$

Now $\frac{nk}{q} > n$, while $k(a+1) \geq n$ by (23). So $k(a+1) > 2n - \frac{nk}{q}$ as required.

The inequality (26) is therefore equivalent to

$$(n-q(a+1))^2+4q(k-q) < (n+q(a+1))^2 - \frac{4qn}{k}(n+q(a+1)) + \frac{4q^2n^2}{k^2},$$

i. e.

$$\begin{aligned} 4q(k-q) &< 4nq(a+1) - \frac{4qn}{k}(n+q(a+1)) + \frac{4q^2n^2}{k^2} \\ &= 4q(k-q)\left(\frac{n}{k}(a+1) - \frac{n^2}{k^2}\right). \end{aligned}$$

Dividing by $4q(k-q)$ and rearranging, this becomes (25). $\square$

In view of Lemma 4.6, we now need to deal only with the cases $ka = n - k + \frac{k^2-1}{n}$ and $ka = n-k+\frac{k^2}{n}$. The second case is impossible since $n > k$. It remains to consider the case

$$ka = n - k + \frac{k^2-1}{n}, \tag{27}$$

which gives

$$n - q(a+1) = n - \frac{qn}{k} - \frac{q(k^2-1)}{kn}. \tag{28}$$

Lemma 4.7. *Suppose* (28) *holds and* $1 \leq q \leq k-1$. *Then*

$$(n-q(a+1))^2 + 4q(k-q) < \left(n - \frac{qn}{k} + \frac{q(k^2-1)}{kn} + \frac{2}{k}\right)^2.$$

Proof. We need to show that

$$4q(k-q) < \left(n - \frac{qn}{k} + \frac{q(k^2-1)}{kn} + \frac{2}{k}\right)^2 - \left(n - \frac{qn}{k} - \frac{q(k^2-1)}{kn}\right)^2. \tag{29}$$

Now the right-hand side of (29) is equal to

$$\begin{aligned} &\left(2n - \frac{2qn}{k} + \frac{2}{k}\right)\left(\frac{2q(k^2-1)}{kn} + \frac{2}{k}\right) \\ &\qquad = \frac{4}{k^2}\left((k-q)q(k^2-1) + (k-q)n + \frac{q(k^2-1)}{n} + 1\right). \end{aligned}$$

So we need to show that

$$0 < -q(k-q) + n(k-q) + \frac{q(k^2-1)}{n} + 1,$$

which is true since $n > k > q$. $\square$

Suppose now that (27) holds. Then Lemmas 4.3 and 4.7 imply that

$$\begin{aligned} R(n,a,k,q) &\leq \left\lfloor \frac{1}{2}\left(n - \frac{qn}{k} - \frac{q(k^2-1)}{kn} + n - \frac{qn}{k} + \frac{q(k^2-1)}{kn} + \frac{2}{k} \right) \right\rfloor \\ &= \left\lfloor n - \frac{qn}{k} + \frac{1}{k} \right\rfloor. \end{aligned}$$

Note that, if $n - \frac{qn}{k} + \frac{1}{k}$ is an integer, then Lemma 4.7 implies that this inequality is strict. Hence in all cases

$$R(n,a,k,q) \leq n - \frac{qn}{k}.$$

Finally $\gcd(n,k) = 1$ by (27) and $0 < q < k$, so $\frac{qn}{k}$ is not an integer. Hence (24) holds. This completes the proof of the theorem.

□

5. The general case

We now start on the computations of C_{12} and C_{21}, where we continue to assume that $0 < k < n$. With the notation of sections 1 and 2, note that, by [1, Lemma 6.5], the flip loci at any critical value can be constructed using only those critical data sets for which there exist (E_1, V_1) and (E_2, V_2) which are both α-stable either for $\alpha = \alpha_c^-$ or for $\alpha = \alpha_c^+$. Since we prefer to have purely numerical conditions on our critical data sets, we shall say that $A_c = (\alpha_c, n_1, d_1, k_1, n_2, d_2, k_2)$ is *allowable* if the numerical conditions (4) and (10) hold together with the Brill-Noether conditions

$$d \geq \frac{1}{k}(n^2-1) - (n-k), \quad d_1 \geq \frac{1}{k_1}(n_1^2 - 1) - (n_1 - k_1) \tag{30}$$

and

$$\text{either } k_2 = 0, n_2 = 1 \text{ or } k_2 \geq 1, d_2 \geq \frac{1}{k_2}(n_2^2-1) - (n_2 - k_2). \tag{31}$$

Proposition 5.1. *Let A_c be an allowable critical data set with $k_2 = 0$. Then $C_{12} > 0$ and $C_{21} > 0$.*

Proof. By (31), we have $n_2 = 1$ and thus $n_1 = n - 1$. Now (9) and (11) imply

$$\alpha_c = \frac{ne + t}{k} \quad \text{with} \quad 0 < e < l.$$

So by (6) and (8)

$$\begin{aligned} C_{12} &= -(n-1) - ne - t + k(a + e + 1) \\ &= ka - (n-k)e - t - n + k + 1 \\ &= l(n-k) + t + m - (n-k)e - t - n + k + 1 \\ &= (l - e - 1)(n-k) + m + 1 > 0 \end{aligned}$$

and by (7) and (8)

$$C_{21} = -(n-1) + ne + t > 0.$$

□

Corollary 5.2. *If $G(\alpha; n, d, 1)$ is non-empty for some α with $t < \alpha < \frac{d}{n-1} - \frac{mn}{n-1}$, then it is non-empty for all such α.*

Proof. For $k = 1$, (4) implies that $k_2 = 0$ for all critical data sets. Hence Proposition 5.1 and Corollary 3.3 imply the assertion. □

This was proved by a different method in [2].

Another case that can be handled easily is when $k_1 \geq n_1$.

Proposition 5.3. *$C_{12} > 0$ for any allowable critical data set with $k_1 \geq n_1$.*

Proof. By (6)

$$C_{12} = (k_1 - n_1)(n_2 + d_2) + d_1 n_2 - k_1 k_2.$$

Now $k < n$ implies $k_2 < n_2$, so

$$C_{12} > (k_1 - n_1)(n_2 + d_2) + (d_1 - k_1) n_2.$$

Hence it suffices to show that $d_1 \geq n_1$, since then

$$C_{12} > (k_1 - n_1)(n_2 + d_2) + (n_1 - k_1) n_2 = d_2(k_1 - n_1),$$

which is non-negative, since $d_2 > 0$ by (4).

In order to see that $d_1 \geq n_1$, suppose first that $k_1 = n_1 + \nu$ with $\nu \geq 1$. Then (30) implies that

$$d_1 \geq \frac{1}{n_1 + \nu}(n_1^2 - 1) + \nu \geq n_1.$$

If $n_1 = k_1 \geq 2$, the same result gives $d_1 \geq n_1 - \frac{1}{n_1}$ which implies the assertion, since d_1 is an integer. Finally, if $n_1 = k_1 = 1$ and $d_1 < 1$, then $d_1 = 0$, which implies $\alpha_c = \frac{d}{n-k}$. This contradicts (10). □

In view of these propositions, we now assume that $k_2 \geq 1$ and $k_1 < n_1$. For this case, we need to rearrange the formula for C_{12}. We have, using (8), (9) and (12),

$$\begin{aligned} C_{12} &= -n_1 n_2 - (ne + n_2 t) + k_1(n_2 a + e) + k_1 n_2 - k_1 k_2 \\ &= -n_1 n_2 - (n - k_1)\left(-\frac{k_2}{k} t + l\left(n_2 - \frac{k_2}{k} n\right) - \frac{f}{k}\right) - n_2 t \\ &\quad + \frac{k_1 n_2}{k}(l(n-k) + m + t) + k_1 n_2 - k_1 k_2. \end{aligned}$$

Hence

$$\begin{aligned} kC_{12} &= l[-(n-k_1)(n_2 k - n k_2) + k_1 n_2 (n-k)] \\ &\quad + t[(n-k_1)k_2 - n_2 k + k_1 n_2] \\ &\quad + (n-k_1) f + k_1 n_2 m + k(k_1 n_2 - k_1 k_2 - n_1 n_2) \\ &= n k_2 (n_1 - k_1) l + k_2 (n_1 - k_1) t + (n - k_1) f \\ &\quad + k_1 n_2 m + k(k_1 n_2 - k_1 k_2 - n_1 n_2) \end{aligned}$$

and thus

$$kC_{12} = (n_1 - k_1)(nk_2l + k_2t - kn_2) + (n - k_1)f + k_1n_2m - kk_1k_2. \tag{32}$$

We now use the assumption $k_2 \geq 1$. The condition (31) is equivalent by (8) and (12) to

$$\begin{aligned} k(n_2a + e) &= n_2(l(n-k) + m + t) - k_2t + kn_2l - nk_2l - f \\ &\geq \frac{k}{k_2}(n_2^2 - 1) - k(n_2 - k_2) \end{aligned}$$

and thus to

$$n_2m \geq -(n_2 - k_2)(ln + t + k) + f + \frac{k}{k_2}(n_2^2 - 1). \tag{33}$$

We can now prove a partial result for $k_1 < n_1$, which will be sufficient for our purposes.

Lemma 5.4. *$C_{12} > 0$ for any allowable critical data set with $k_1 < n_1$, $kk_1 < nk_2$ (resp. $kk_1 \leq nk_2$) and $nk_2l + k_2t - kn_2 \leq 0$ (resp. $nk_2l + k_2t - kn_2 < 0$).*

Proof. Inserting (33) in (32), we get

$$\begin{aligned} kC_{12} &\geq (n_1 - k_1)(nk_2l + k_2t - kn_2) + (n - k_1)f - kk_1k_2 \\ &\quad -(n_2 - k_2)k_1(ln + t + k) + k_1f + \frac{kk_1}{k_2}(n_2^2 - 1) \\ &= (k_2(n_1 - k_1) - k_1(n_2 - k_2))(nl + t) - kn_2(n_1 - k_1) \\ &\quad -kk_1(n_2 - k_2) - kk_1k_2 + \frac{kk_1}{k_2}(n_2^2 - 1) + (n - k_1)f + k_1f \end{aligned}$$

i. e.

$$kk_2C_{12} \geq (k_2n_1 - k_1n_2)(nk_2l + k_2t - kn_2) + k_2nf - kk_1. \tag{34}$$

Note that $k_2n_1 - k_1n_2 < 0$ and $f \geq 1$. The result follows. □

The formulae (32) and (34) are complementary to one another in that the first is of use when $nk_2l + k_2t - kn_2 \geq 0$ and the second when $nk_2l + k_2t - kn_2 \leq 0$. This is sufficient to handle another special case.

Proposition 5.5. *Let A_c be an allowable critical data set with $k_1 = 1$. Then $C_{12} > 0$.*

Proof. If $k_2 = 0$, this follows from Proposition 5.1, while, if $n_1 = 1$, it follows from Proposition 5.3. If $n_1 \geq 2$, $k_2 \geq 1$ and $nk_2l + k_2t - kn_2 \leq 0$, then we have $kk_1 = k < n \leq nk_2$ and the result follows from Lemma 5.4.

If $nk_2l + k_2t - kn_2 > 0$, then (32) gives

$$kC_{12} \geq (n_1 - 1) + (n - 1)f + n_2m - kk_2.$$

From (13) we get $f \equiv t + nl \equiv -m \mod k$; moreover $f \geq 1$. If $0 \leq m \leq k-1$, then $f \geq k - m$ and

$$kC_{12} \geq (n_1-1)+(n-1)(k-m)+n_2m-kk_2 = n_1-1+(n-k)k-(n_1-1)m.$$

But $m < k$ and $n_1 - 1 < n - 1 - k_2 = n - k$, since $n_2 > k_2$ by (4). So $kC_{12} > 0$.

Finally, if $m \geq k$, then $kC_{12} \geq n_2 k - kk_2 > 0$. □

We now turn to look at C_{21}.

Lemma 5.6. *Suppose $k_2 \geq 1$. Then $C_{21} > 0$ in each of the following cases:*

(i) *$e \geq 1$, $k_2 \geq 2$, $n \geq k_2(k_1+1)$,*
(ii) *$e \geq 1$, $k_2 = 1$, $n \geq 2k_1 + 1$,*
(iii) *$e \leq 0$, $n \geq k(1 + k_1k_2)$.*

Proof. Substituting $d_1 = d - d_2$ in (7) and using (8),

$$\begin{aligned} C_{21} &= -n_1n_2 + d_2(n-k_2) - d(n_2-k_2) + k_2(n_1-k_1) \\ &= -n_1n_2 + (n_2a+e)(n-k_2) - (na-t)(n_2-k_2) + k_2(n_1-k_1) \\ &= n_1(k_2(a+1)-n_2) + e(n-k_2) + t(n_2-k_2) - k_1k_2 \qquad (35) \end{aligned}$$

By (8) and (31), we have

$$k_2(a+1) - n_2 \geq \frac{k_2^2 - 1 - k_2e}{n_2}.$$

So

$$\begin{aligned} C_{21} &\geq \frac{n_1(k_2^2-1-k_2e)}{n_2} + e(n-k_2) + t(n_2-k_2) - k_1k_2 \\ &= (n_2-k_2)\left(\frac{ne}{n_2} + t\right) + \frac{n_1(k_2^2-1)}{n_2} - k_1k_2 \qquad (36) \end{aligned}$$

If $e \geq 1$, this gives

$$\begin{aligned} C_{21} &\geq (n_2-k_2)\frac{n}{n_2} + \frac{n_1(k_2^2-1)}{n_2} - k_1k_2 \\ &= n - k_2(k_1+1) + \frac{n_1(k_2^2-k_2-1)}{n_2}. \end{aligned}$$

So $C_{21} > 0$ if $n \geq k_2(k_1+1)$ and $k_2 \geq 2$, proving (i). If $k_2 = 1$, (4) gives $\frac{n_1}{n_2} < k_1$, so

$$C_{21} \geq n - (k_1+1) - \frac{n_1}{n_2} > n - 2k_1 - 1.$$

So $C_{21} > 0$ if $n \geq 2k_1 + 1$, proving (ii).

If $e \le 0$, then $\frac{ne}{n_2} \ge \frac{ke}{k_2}$, so $\frac{ne}{n_2} + t \ge \frac{ke}{k_2} + t \ge \frac{1}{k_2}$ by (11). So (36) gives

$$
\begin{aligned}
C_{21} &\ge \frac{n_2}{k_2} - 1 + \frac{n_1(k_2^2-1)}{n_2} - k_1k_2 \\
&> \frac{n}{k} + \frac{n_1(k_2^2-1)}{n_2} - (1+k_1k_2).
\end{aligned} \tag{37}
$$

So $C_{21} > 0$ if $n \ge k(1+k_1k_2)$, proving (iii). □

Remark 5.7. If $k_2 = 1$, $e \le 0$, then (37) gives $C_{21} \ge n_2 - (k_1+1) = n_2 - k$, with equality possible only if $e = 0$, $t = 1$.

These results are not sufficient for us to determine precisely when $C_{12} > 0$ or $C_{21} > 0$. We shall see in sections 7 and 10 that both C_{12} and C_{21} can be 0. However we can now prove

Theorem 5.8. *Let k be a fixed positive integer. Then there are only finitely many allowable critical data sets with $n > k$ for which $C_{12} \le 0$ or $C_{21} \le 0$.*

Proof. By Proposition 5.1, we can suppose that $k_2 \ge 1$.

Combining Proposition 5.3 with Lemma 5.4, we see that $C_{12} > 0$ when $n > \frac{kk_1}{k_2}$, except possibly when

$$k_1 < n_1 \quad \text{and} \quad nk_2l + k_2t - kn_2 > 0.$$

In this case we apply (32). Since $f \ge 1$ and $m \ge 0$, we get

$$kC_{12} > n - k_1 - kk_1k_2.$$

So $C_{12} > 0$ if $n \ge k_1 + kk_1k_2$. It remains to show that, if we fix n as well as k, then $C_{12} > 0$ for all but finitely many values of d. In view of Proposition 5.3, we need only prove this when $k_1 < n_1$. In this case, it follows immediately from (32) that $C_{12} > 0$ for all sufficiently large values of l, say $l \ge A$. But it follows easily from the definition of l that this certainly holds if

$$kd \ge (n-k)((A+1)n - 1).$$

Turning to C_{21}, it follows at once from Lemma 5.6 that, for any fixed k, $C_{21} > 0$ for all sufficiently large n. If we fix n as well as k, and insert $e > -\frac{k_2}{k}t$ in (35), we obtain

$$C_{21} > n_1(k_2(a+1) - n_2) + t\left(n_2 - k_2 - \frac{nk_2}{k} + \frac{k_2^2}{k}\right) - k_1k_2.$$

So $C_{21} > 0$ for all sufficiently large values of a and hence for all but finitely many values of d. □

Corollary 5.9. *Let k be a fixed positive integer. Then, for all but finitely many pairs (n, d) with $n > k$, one of the following two possibilities holds:*

- $G(\alpha; n, d, k) = \emptyset$ *for all α;*

- *$G(\alpha; n, d, k) \neq \emptyset$ for all α such that*
$$\frac{t}{k} < \alpha < \frac{ln + t}{k}.$$

Proof. This follows from the theorem, Corollary 3.3 and (10). □

When $t = 0$, we have a stronger result.

Corollary 5.10. *Let k be a fixed positive integer. Then, for all but finitely many pairs (n, a) such that $n > k$ and (23) holds, the moduli space $G(\alpha; n, na, k) \neq \emptyset$ if and only if*
$$0 < \alpha < \frac{ln}{k}.$$

Proof. This follows from Corollary 5.9 and Theorem 4.5. □

We finish this section by showing how we can use these results to construct α-stable coherent systems for certain values of $t > 0$.

We begin with a lemma

Lemma 5.11. *Suppose that $t \geq 1$, $ka \geq n - k + t$ and (E_1, W) is a coherent system of type $(t, t(a-1), k)$. Then*
$$\dim \operatorname{Ext}^1((E_1, W), (\mathcal{O}(a), 0)) \geq n - t.$$

Proof. By (6) and [1, equation (8)],
$$\begin{aligned} \dim \operatorname{Ext}^1((E_1, W), (\mathcal{O}(a), 0)) &\geq -t - at + t(a-1) + k(a+1) \\ &= -2t + k(a+1) \\ &\geq -2t + k + n - k + t = n - t. \end{aligned}$$
□

Proposition 5.12. *Suppose $k \geq 2$, $ka \geq n - k + t$ and that one of the following four conditions holds:*

- *$t = 1$ and $a \geq k$;*
- *$t = k - 1$ and $a \geq 2$;*
- *$t = k$ and $a \geq 3$;*
- *$t > k$, $ka \geq t + \frac{k^2-1}{t}$ and $C_{12} > 0$ for all allowable critical data sets for coherent systems of type $(t, t(a-1), k)$.*

Then
$$G((t/k)^+; n, d, k) \neq \emptyset.$$

Proof. We show first that the hypotheses imply that
$$G(t/k; t, t(a-1), k) \neq \emptyset.$$

For $t = 1$, we require only the condition $h^0(\mathcal{O}(a-1)) \geq k$, which is equivalent to $a \geq k$.

For $t = k - 1$, the result follows from [2, proposition 6.4].

For $t = k$, it follows from [2, Proposition 6.3] that
$$G(\tilde{\alpha}; t, t(a-1), t) \neq \emptyset$$

for some $\tilde{\alpha} > 0$ if and only if $a \geq 3$. Taking a general element (E, V) of this moduli space, we can assume by [2, Theorem 3.2 and Proposition 3.6] that $E = \mathcal{O}(a-1)^t$ and that V generically generates $\mathcal{O}(a-1)^t$. If (F, W) is a coherent subsystem of (E, V) which contradicts α-stability for some $\alpha > 0$, then we must have $F = \mathcal{O}(a-1)^r$, $\dim W = r$ for some r, $0 < r < t$. But then (F, W) contradicts α-stability for all $\alpha > 0$ and in particular for $\alpha = \tilde{\alpha}$. This is a contradiction, establishing that (E, V) is α-stable for all $\alpha > 0$.

Finally, if $t > k$, the hypothesis on the allowable critical data sets implies, by Theorem 4.5 and Corollary 3.3, that $G(t/k; t, t(a-1), k) \neq \emptyset$ provided that

$$0 < \frac{t}{k} < \frac{t(a-1)}{t-k} - \frac{m't}{k(t-k)}$$

for a certain integer m' with $0 \leq m' < t-k$. This condition is equivalent to

$$t(t-k) < kt(a-1) - m't,$$

i. e. $ka > t + m'$. But $m' < t - k < n - k$, so this follows from the hypothesis $ka \geq n - k + t$.

We now consider extensions

$$0 \to (\mathcal{O}(a)^{n-t}, 0) \to (E, V) \to (E_1, W) \to 0,$$

where (E_1, W) is a t/k-stable coherent system of type $(t, t(a-1), k)$. Note that

$$\mu_{t/k}(E_1, W) = a - 1 + \frac{t}{k} \cdot \frac{k}{t} = a.$$

By Lemmas 3.9 and 5.11, the general extension of this form is $(t/k)^+$-stable. This completes the proof. □

Corollary 5.13. *Let k be a fixed integer, $k \geq 2$. For all but a finite number of pairs (n, d) for which $n > k$, $ka \geq n - k + t$ and one of the conditions*

- $t = 1$ *and* $a \geq k$,
- $t = k - 1$ *and* $a \geq 2$,
- $t = k$ *and* $a \geq 3$,
- $t > k$ *and* $ka \geq t + \frac{k^2-1}{t}$

holds, the moduli space $G(\alpha; n, d, k) \neq \emptyset$ if and only if

$$\frac{t}{k} < \alpha < \frac{d}{n-k} - \frac{mn}{k(n-k)}.$$

Proof. In view of Theorem 5.8, we can assume that $C_{12} > 0$ for all allowable critical data sets for coherent systems of type (n, d, k). In the case $t > k$, a given pair (t, a) can arise from only finitely many pairs (n, d) which satisfy the condition $ka \geq n - k + t$. We can therefore also assume that $C_{12} > 0$ for all allowable critical data sets for coherent systems of type $(t, t(a-1), k)$. The proposition now implies that $G((t/k)^+; n, d, k) \neq \emptyset$ and the result follows from Corollary 3.3. □

6. The case $k = 2$

In the case $k = 2$, we can use the methods developed above to give a simpler proof of [2, Theorem 5.4]. Note first that it follows from Propositions 5.1 and 5.5 that $C_{12} > 0$ for any allowable critical data set and that $C_{21} > 0$ except possibly when $k_1 = k_2 = 1$.

Lemma 6.1. *Let $n \geq 3$ and let A_c be an allowable critical data set with $k_1 = k_2 = 1$. Then $C_{21} > 0$.*

Proof. If $e \geq 1$, this follows at once from Lemma 5.6(ii).

If $e \leq 0$, Remark 5.7 gives $C_{21} \geq n_2 - 2$, with equality possible only if $e = 0$ and $t = 1$. Now $n_2 \geq 2$ by (4). Hence $C_{21} > 0$ except possibly when $e = 0$, $t = 1$, $n_2 = 2$, and then $n_1 = 1$ by (4). Moreover $d = 3a-1$ and (30) implies that $d \geq 3$. Hence $a \geq 2$ and by (7) and (8)

$$C_{21} = -2 + 2a - 2(a-1) + (a-1+1-1) = a - 1 \geq 1,$$

which completes the proof of the lemma. □

Corollary 6.2. *If $G(\alpha; n, d, 2)$ is non-empty for some α with $\frac{t}{2} < \alpha < \frac{d}{n-2} - \frac{mn}{2(n-2)}$, then it is non-empty for all such α.*

Proof. It suffices to show that C_{12} and C_{21} are both positive for all allowable critical data sets A_c. But for $k = 2$ either $k_2 = 0$ or $k_1 = k_2 = 1$. Hence Propositions 5.1 and 5.5 and Lemma 6.1 imply the assertion. □

For the proof of the full result of [2] for $k = 2$, it remains to determine when there exists an α-stable coherent system of type $(n, d, 2)$ for some α.

Proposition 6.3. *Suppose $n \geq 3$, $l \geq 1$, $d \geq \frac{1}{2}n(n-2) + \frac{3}{2}$ and $(n, d) \neq (4, 6)$. Then there exists a $(t/2)^+$-stable coherent system (E, V) of type $(n, d, 2)$.*

Proof. For $t = 0$, this has already been proved in Theorem 4.5.

For $t \geq 1$, it is sufficient to verify that the conditions of Proposition 5.12 are satisfied. Note that the hypothesis $l \geq 1$ is equivalent to

$$2a \geq n - 2 + t, \tag{38}$$

while the Brill-Noether condition $d \geq \frac{1}{2}(n^2 - 1) - (n-2)$ is easily seen to be equivalent to

$$2a \geq n - 2 + \frac{3 + 2t}{n}. \tag{39}$$

For $t = 1$, (39) gives $2a \geq n - 2 + \frac{5}{n}$, which implies $a \geq 2$ as required. For $t \geq 3$, the condition $2a \geq t + \frac{3}{t}$ follows from (38), while $C_{12} > 0$ holds always for $k = 2$. For $t = 2$, (38) gives $2a \geq n$, which implies $a \geq 3$ as required except for $n = 3, 4$. We are left therefore with the two cases $(n, d) = (3, 4)$ and $(n, d) = (4, 6)$, for both of which $a = 2$.

The case $(n,d)=(4,6)$ has been excluded in the statement, so we need only to prove the proposition for $(n,d)=(3,4)$.

In this case, we have $a=t=2$. The moduli space $G(1;2,2,2)$ is empty by [2, Proposition 5.6], but there do exist 1-semistable coherent systems of type $(2,2,2)$, which have the form

$$(E_1,W)=(\mathcal{O}(1),W_1)\oplus(\mathcal{O}(1),W_2).$$

Since $h^0(\mathcal{O}(1))=2$, we can take W_1 and W_2 to be distinct subspaces of $H^0(\mathcal{O}(1))$ of dimension 1. Let (E,V) be the general extension of the form

$$0\to(\mathcal{O}(2),0)\to(E,V)\to(E_1,W)\to 0.$$

Comparing this with (18), it is easy to verify (20). It follows from Lemma 3.9 that (E,V) is 1^+-stable as required. □

We can now restate [2, Theorem 5.4].

Theorem 6.4. *Suppose $n\geq 3$. Then $G(\alpha;n,d,2)\neq\emptyset$ for some α if and only if $l\geq 1$, $d\geq\frac{1}{2}n(n-2)+\frac{3}{2}$ and $(n,d)\neq(4,6)$. Moreover, when these conditions hold, $G(\alpha;n,d,k)\neq\emptyset$ if and only if*

$$\frac{t}{2}<\alpha<\frac{d}{n-2}-\frac{mn}{2(n-2)}.$$

Proof. The stated conditions are sufficient by Proposition 6.3. Conversely, if $G(\alpha;n,d,k)\neq\emptyset$, then $l\geq 1$ and $d\geq\frac{1}{2}n(n-2)+\frac{3}{2}$ by [2, Remark 4.3 and Corollary 3.3]. It is easy to prove that there do not exist α-stable coherent systems of type $(4,6,2)$ (see the first paragraph of the proof of [2, Theorem 5.4]). For the last part, see Corollary 6.2. □

7. The case $k=3$

Now suppose $k=3$. In this section we will show that C_{12} is positive for all allowable critical data sets A_c and determine those A_c for which $C_{21}=0$. As a consequence, we give examples for which the lower bound of [2, Proposition 4.1] for those α, for which there exist α-stable systems, is not best possible.

Proposition 7.1. *Let $n\geq 4$. Suppose A_c is an allowable critical data set with $k=3$. Then $C_{12}>0$.*

Proof. The cases $k_1=3$ and $k_1=1$ are covered by Propositions 5.1 and 5.5. So suppose $k_1=2,\ k_2=1$. Then (4) implies

$$n_1<2n_2,$$

and by (13)

$$f\equiv m \bmod 3.$$

According to (32)

$$3C_{12}=(n_1-2)(nl+t-3n_2)+(n-2)f+2n_2m-6.$$

If $n_1 \leq 2$, C_{12} is positive by Proposition 5.3. So let $n_1 \geq 3$. If $nl + t - 3n_2 \geq 0$, then $C_{12} > 0$, since $n_2 \geq 2$ and thus $n \geq 5$ and either f and m are both positive or $f \geq 3$.

If $nl + t - 3n_2 < 0$, then, using (34), we get

$$3C_{12} \geq (2n_2 - n_1)(3n_2 - nl - t) + nf - 6,$$

which is positive for $n \geq 6$. For $n = 5$, we have $n_1 = 3,\ n_2 = 2$ and

$$3C_{12} = 5l + t + 3f + 4m - 12.$$

This is ≥ 0 since $l > 0$ and either f and m are both positive or $f \geq 3$. Equality occurs if and only if $l = f = m = 1,\ t = 0$. But then (12) gives $e = 0$, which contradicts (11). □

Lemma 7.2. *Let $n \geq 4$. Suppose A_c is an allowable critical data set with $k_1 = 1, k_2 = 2$. Then $C_{21} > 0$.*

Proof. For $e \geq 1$, this follows at once from Lemma 5.6(i).

For $e \leq 0$, we need to analyse (36) and (37) more carefully. In our case (37) becomes

$$C_{21} \geq \frac{n_2}{2} + 3\frac{n_1}{n_2} - 3.$$

This is positive if $n_2 \geq 5$. For $n_2 = 4$, we note that (4) implies that $n = 5,\ n_1 = 1$. By (11), we have $e \geq -\frac{2}{3}t + \frac{1}{3}$, so by (36)

$$C_{21} \geq 2\left(\frac{5e}{4} + t\right) + \frac{3}{4} - 2 = \frac{5e}{2} + 2t - \frac{5}{4} \geq \frac{t}{3} + \frac{5}{6} - \frac{5}{4}.$$

This is positive if $t \geq 2$. Since $e \leq 0$, the only remaining case is when $t = 1$, which implies $e \geq 0$, so $C_{21} \geq 2t - \frac{5}{4} > 0$.

It remains to consider the case $e \leq 0,\ n_2 = 3$. In this case, (4) gives $n = 4,\ n_1 = 1$. We now have by (35)

$$C_{21} = 2(a+1) - 3 + 2e + t - 2 = -3 + 2a + t + 2e.$$

Since $e \leq 0$, we have $t \geq 1$. Moreover (30) gives $d \geq 4$, so $a \geq 2$. If $t = 1$, then $e = 0$, while, if $t = 2$ or $t = 3$, then $e \geq -1$. So in all cases $C_{21} > 0$. □

Proposition 7.3. *Suppose $n \geq 4$ and A_c is an allowable critical data set with $k_1 = 2, k_2 = 1$. Then $C_{21} > 0$, except in the cases*

(a): $(n_1, n_2, d_1, d_2) = (4, 3, 7, 6)$, $\quad \alpha_c = \frac{3}{2}$,
(b): $(n_1, n_2, d_1, d_2) = (3, 3, 5, 6)$, $\quad \alpha_c = 1$,
(c): $(n_1, n_2, d_1, d_2) = (2, 3, 3, 6)$, $\quad \alpha_c = \frac{3}{4}$,
(d): $(n_1, n_2, d_1, d_2) = (1, 3, 1, 6)$, $\quad \alpha_c = \frac{3}{5}$,

where $C_{21} = 0$.

Proof. In this case (4) gives

(40) $$n < 3n_2;$$

since $n \geq 4$, this implies that $n_2 \geq 2$. By (36) we have

$$C_{21} \geq (n_2 - 1)(\frac{ne}{n_2} + t) - 2. \tag{41}$$

We distinguish several cases:

Case 1: $n_2 = 2$. According to (40), $n_1 = 2$ or 3. Suppose first $(n_1, n_2) = (2, 2)$. By (7) we have $C_{21} = 2d_2 - d_1 - 4$. By (30), $d_1 \geq 2$ and (4) implies that $d_2 \geq d_1 + 1$. Hence, if $d_1 \geq 3$ or $d_1 = 2, d_2 \geq 4$, we have $C_{21} > 0$. In the remaining case $(d_1, d_2) = (2, 3)$, we have $t = 3$, $e = -1$, so (11) fails.

Now suppose $(n_1, n_2) = (3, 2)$. Then $C_{21} = 3d_2 - d_1 - 5$. By (30), $d_1 \geq 3$ and (4) implies $d_2 > \frac{2}{3}d_1$. Hence, if $d_1 \geq 5$ or $d_1 = 3, d_2 \geq 3$ or $d_1 = 4, d_2 \geq 4$, we have $C_{21} > 0$. In the remaining case $(d_1, d_2) = (4, 3)$, we have $t = 3$, $e = -1$, so again (11) fails.

Case 2: $e \geq 1$, $n_2 \geq 3$. (41) gives $C_{21} \geq (n_2 - 1)(1 + \frac{n_1}{n_2}) - 2 > 0$.

Case 3: $e \leq 0$ $n_2 \geq 3$, $(n_2, t, e) \neq (3, 1, 0)$. The result follows from Remark 5.7.

Case 4: $e = 0$, $t = 1$, $n_2 = 3$. By (40) we have $1 \leq n_1 \leq 5$. Moreover $d_2 = 3a$ by (8) and hence $d_1 = n_1 a - t = n_1 a - 1$. So, by (7),

$$C_{21} = n_1 d_2 - 2d_1 - 2(n_1 + 1) = n_1(a - 2).$$

By (31), $d_2 \geq 6$, so $a \geq 2$ and $C_{21} \geq 0$. Now the Brill-Noether inequality $d \geq \frac{1}{3}(n^2 - 1) - (n - 3)$ gives

$$3a \geq n - 3 + \frac{8 + 3t}{n} = n - 3 + \frac{11}{n}. \tag{42}$$

Using this, we see that $a = 2$ only in the four cases listed (note that $n_1 = 5$ does not occur, since then (42) gives $a \geq 3$). One can easily compute α_c in each of the exceptional cases and check that (10) holds. $\square$

Proposition 7.4. *For all cases other than those covered by Proposition 7.3 (a)–(d), if $G(\alpha; n, d, 3)$ is non-empty for some α with $\frac{t}{3} < \alpha < \frac{d}{n-3} - \frac{mn}{3(n-3)}$, then it is non-empty for all such α.*

Proof. This follows from Propositions 5.1 and 7.1, Lemma 7.2 and Proposition 7.3, together with Corollary 3.3. $\square$

8. Existence for $k = 3$

We consider first the existence of α-stable coherent systems in the exceptional cases of Proposition 7.3.

Proposition 8.1. *In each of the following cases, we have $G(\alpha; n, d, 3) = \emptyset$ for $\alpha \leq \alpha_c$ and $G(\alpha_c^+; n, d, 3) \neq \emptyset$:*

(a): $(n,d)=(7,13)$, $\alpha_c=\frac{3}{2}$,
(b): $(n,d)=(6,11)$, $\alpha_c=1$,
(c): $(n,d)=(5,9)$, $\alpha_c=\frac{3}{4}$,
(d): $(n,d)=(4,7)$, $\alpha_c=\frac{3}{5}$.

Proof. In each case $C_{12}>0$ for all allowable critical data sets by Proposition 7.1 and, by Lemma 7.2 and Proposition 7.3, $C_{21}>0$ except for a unique critical data set as given in Proposition 7.3. In view of Corollaries 3.3 and 3.5, and Remarks 3.7 and 3.8, it is therefore sufficient to prove that there exist α_c-stable coherent systems (E_1,V_1) of type $(n_1,d_1,2)$ and (E_2,V_2) of type $(n_2,d_2,1)$, where n_1, n_2, d_1, d_2 are as given in Proposition 7.3.

For (E_2,V_2), we have $(n_2,d_2)=(3,6)$ in every case and, with the obvious notation, $t_2=0$, $m_2=0$. So, by [2, Theorem 5.1], we require

$$0<\alpha_c<\frac{6}{2}=3,$$

which is true in every case.

For (E_1,V_1), in cases (a) and (b) we need to apply Theorem 6.4 (or [2, Theorem 5.4]). Certainly $(n_1,d_1)\neq(4,6)$ and it is easy to check that $l_1\geq 1$ and $d_1\geq\frac{1}{2}n_1(n_1-2)+\frac{3}{2}$; in fact the latter was one of the conditions for an allowable critical data set. It remains to prove that in each case

$$\frac{t_1}{2}<\alpha_c<\frac{d_1}{n_1-2}-\frac{m_1n_1}{2(n_1-2)}$$

In fact the numbers in each case are given by

(a): $n_1=4$, $d_1=7$, $t_1=1$, $m_1=1$,
(b): $n_1=3$, $d_1=5$, $t_1=1$, $m_1=0$,

and the result is clear.

In case (c), we have $n_1=k_1=2$, $d_1=3$, so the result follows from [2, Proposition 5.6]. Finally, in case (d), we have $n_1=1$, $k_1=2$, $d_1=1$, so $(E_1,V_1)\simeq(\mathcal{O}(1),H^0(\mathcal{O}(1)))$ is α-stable for all $\alpha>0$. $\square$

We turn now to the general case.

Proposition 8.2. *Suppose $n\geq 4$, $l\geq 1$, $d\geq\frac{1}{3}n(n-3)+\frac{8}{3}$ and*

$$(n,d)\neq(7,13),\ (6,11),\ (6,9),\ (5,9),\ (4,7).$$

Then there exists a $(t/3)^+$-stable coherent system of type $(n,d,3)$.

Proof. For $t=0$, this has already been proved in Theorem 4.5. For $t\geq 1$, it is sufficient to verify that the conditions of Proposition 5.12 are satisfied.

Note first that the condition

$$3a\geq n-3+t \tag{43}$$

is equivalent to $l\geq 1$.

For $t = 1$, we require $a \geq 3$. By (42), the only cases for which $a < 3$ are when $a = 2$ and $4 \leq n \leq 7$, giving rise precisely to the exceptional cases $(7, 13)$, $(6, 11)$, $(5, 9)$ and $(4, 7)$.

For $t = 2$, we require $a \geq 2$, which follows at once from (42).

For $t = 3$, we require again $a \geq 3$. By (42), we have

$$3a \geq n - 3 + \frac{17}{n},$$

which implies $a \geq 3$ except in the cases

$$(n, d) = (6, 9),\ (5, 7),\ (4, 5).$$

For $(n, d) = (5, 7)$ and $(n, d) = (4, 5)$, we consider extensions

$$0 \to (\mathcal{O}(2)^{n-3}, 0) \to (E, V) \to \bigoplus_{i=1}^{3} (\mathcal{O}(1), W_i) \to 0,$$

where the W_i are distinct subspaces of $H^0(\mathcal{O}(1))$ of dimension 1. We need to check the inequalities (20), which in this case give $3 > n - 3$ and $3 \geq 2$. These are valid, so Lemma 3.9 establishes that the general (E, V) is α_c^+-stable, where here $\alpha_c = 1 = \frac{t}{3}$.

Finally, for $t \geq 4$, we certainly have $C_{12} > 0$ for all allowable critical data sets for coherent systems of type $(t, t(a-1), 3)$ by Proposition 7.1. The condition $3a \geq t + \frac{8}{t}$ follows from (43) since we now have $n \geq 5$.

This completes the proof. $\square$

Remark 8.3. The construction fails for $(n, d) = (6, 9)$ because we no longer have $3 > n - 3$. In fact, it is easy to see that $G(1^+; 6, 9, 3) = \emptyset$. Indeed, if this is not so, then a general element of $G(1^+; 6, 9, 3)$ has $E \simeq \mathcal{O}(2)^3 \oplus \mathcal{O}(1)^3$. Now E has a unique subbundle $F \simeq \mathcal{O}(2)^3$. If $V_1 = V \cap H^0(F) \neq 0$, then (F, V_1) is a coherent subsystem of (E, V) which contradicts 1^+-stability. So there is an exact sequence

$$0 \to (\mathcal{O}(2)^3, 0) \to (E, V) \to (\mathcal{O}(1)^3, W) \to 0 \tag{44}$$

with $\dim W = 3$. The homomorphism $W \otimes \mathcal{O} \to \mathcal{O}(1)^3$ is not an isomorphism, so there exists a section of $\mathcal{O}(1)^3$ contained in W and possessing a zero. This defines a coherent subsystem $(\mathcal{O}(1), W_1)$ of $(\mathcal{O}(1)^3, W)$ with $\dim W_1 = 1$. Now consider the pullback

$$0 \to (\mathcal{O}(2)^3, 0) \to (E_1, V_1) \to (\mathcal{O}(1), W_1) \to 0 \tag{45}$$

of (44). Such extensions are classified by triples (e_1, e_2, e_3) with $e_i \in \mathrm{Ext}^1((\mathcal{O}(1), W_1), (\mathcal{O}(2), 0))$. Note that

$$\mathrm{Hom}((\mathcal{O}(1), W_1), (\mathcal{O}(2), 0)) = 0.$$

Hence, from (16) and (6), we see that

$$\dim \mathrm{Ext}^1((\mathcal{O}(1), W_1), (\mathcal{O}(2), 0)) = 1.$$

It follows that, using an automorphism of $\mathcal{O}(2)^3$, we can suppose that $e_2 = e_3 = 0$. This means that (45) is induced from an exact sequence

$$0 \to (\mathcal{O}(2), 0) \to (E_2, V_2) \to (\mathcal{O}(1), W_1) \to 0.$$

But now (E_2, V_2) is a coherent subsystem of (E, V) which contradicts α-stability of (E, V) for all α. Hence $G(1^+; 6, 9, 3) = \emptyset$ as asserted. It follows from Proposition 7.4 that $G(\alpha; 6, 9, 3) = \emptyset$ for all α.

Theorem 8.4. *Suppose $n \geq 4$. Then $G(\alpha; n, d, 3) \neq \emptyset$ for some $\alpha > 0$ if and only if $l \geq 1$, $d \geq \frac{1}{3}n(n-3) + \frac{8}{3}$ and $(n, d) \neq (6, 9)$. Moreover, when these conditions hold, $G(\alpha; n, d, 3) \neq \emptyset$ if and only if*

$$\frac{t}{3} < \alpha < \frac{d}{n-3} - \frac{mn}{3(n-3)},$$

except for the following pairs (n, d), where the range of α is as stated :

$$\begin{array}{llll} for \quad (4,7): & \frac{3}{5} < \alpha < 7; & for \quad (5,9): & \frac{3}{4} < \alpha < \frac{11}{3}; \\ for \quad (6,11): & 1 < \alpha < \frac{7}{3}; & for \quad (7,13): & \frac{3}{2} < \alpha < \frac{8}{3}. \end{array}$$

Proof. The necessity of the conditions follows from [2, Corollary 3.3 and Remark 4.3] and Remark 8.3. Sufficiency has been proved in Propositions 8.1 and 8.2. The assertion about the range of α follows from Proposition 7.4 except for the exceptional cases, when it is a consequence of Propositions 7.1 and 8.1 and Corollary 3.3. □

9. The case $k = 3$, $n \leq 3$

In this section, we will complete the results for $k = 3$ by considering the case $n \leq 3$. It is interesting to note that further exceptional cases arise. We begin with a general result, which completes [2, Proposition 6.3] in the case $t = 0$.

Proposition 9.1. *For any $n \geq 2$, $G(\alpha; n, na, n) \neq \emptyset$ if and only if $a \geq 2$ and $\alpha > 0$.*

Proof. By [2, Proposition 6.3], $G(\alpha; n, na, n) \neq \emptyset$ for some $\alpha > 0$ if and only if $a \geq 2$ and there is then no upper bound on α. In view of [2, Corollary 3.4], it is therefore sufficient to prove that $G(0^+; n, na, n) \neq \emptyset$ if $a \geq 2$. But this is exactly what is shown in the proof of the case $t = k$ of Proposition 5.12. □

Theorem 9.2.

(i): *$G(\alpha; 1, d, 3) \neq \emptyset$ if and only if $d \geq 2$ and $\alpha > 0$.*

(ii): *$G(\alpha; 2, d, 3) \neq \emptyset$ for some α if and only if $d \geq 2$. Moreover, if $d \geq 2$, $G(\alpha; 2, d, 3) \neq \emptyset$ for all $\alpha > \frac{t}{3}$ except in the case $d = 3$, when $G(\alpha; 2, 3, 3) \neq \emptyset$ if and only if $\alpha > 1$.*

(iii): *$G(\alpha; 3, d, 3) \neq \emptyset$ for some α if and only if $d \geq 4$. Moreover, if $d \geq 4$, $G(\alpha; 3, d, 3) \neq \emptyset$ for all $\alpha > \frac{t}{3}$ except in the case $d = 5$, when $G(\alpha; 3, 5, 3) \neq \emptyset$ if and only if $\alpha > \frac{2}{3}$.*

Proof. (i) follows immediately from the fact that $h^0(\mathcal{O}(d)) \geq 3$ if and only if $d \geq 2$.

(ii): By [2, Proposition 6.4], $G(\alpha; 2, d, 3) \neq \emptyset$ for some α if and only if $d \geq 2$ and there is then no upper bound on α. Moreover, if $d \geq 2$ and $t = 0$, then, by the same proposition, $G(\alpha; 2, d, 3) \neq \emptyset$ for all $\alpha > 0$. This completes the proof when $t = 0$.

If $t = 1$, it follows from Proposition 5.12 that $G((1/3)^+; 2, d, 3) \neq \emptyset$ if $a \geq 3$, i. e. $d \geq 5$; the result now follows from [2, Corollary 3.4]. Now suppose $d = 3$ and consider a coherent system $(E, V) = (\mathcal{O}(2) \oplus \mathcal{O}(1), V)$ of type $(2, 3, 3)$ such that V generates E; in fact we can take V to be a general subspace of $H^0(E)$ of dimension 3. If (F, W) is any coherent subsystem of (E, V) with $\mathrm{rk} F = 1$, then $\dim W \leq 1$ (otherwise E/F would not be generated by V/W). Moreover $\deg F \leq 2$, so (E, V) is α-stable provided

$$2 + \alpha < \frac{3}{2} + \frac{3\alpha}{2},$$

i. e. $\alpha > 1$. Conversely, if (E, V) is any α-stable coherent system of type $(2, 3, 3)$, then $E \simeq \mathcal{O}(2) \oplus \mathcal{O}(1)$. Since $h^0(\mathcal{O}(1)) = 2$, (E, V) has a coherent subsystem of type $(1, 2, 1)$, which contradicts α-stability for $\alpha \leq 1$.

(iii): By [2, Proposition 6.3], $G(\alpha; 3, d, 3) \neq \emptyset$ for some α if and only if $d \geq 4$ and there is then no upper bound on α. So suppose that $d \geq 4$. If $t = 0$, Proposition 9.1 gives the result. If $t = 1$, $d \geq 8$ or $t = 2$, Proposition 5.12 implies that $G((t/3)^+; 3, d, 3) \neq \emptyset$, and the result follows from [2, Corollary 3.4].

There remains the case $d = 5$. We consider a coherent system $(E, V) = (\mathcal{O}(2)^2 \oplus \mathcal{O}(1), V)$ of type $(3, 5, 3)$, where we choose V so that V generically generates E, $\dim V \cap H^0(\mathcal{O}(2)^2) = 1$ and the line subbundle generated by a non-zero element of $V \cap H^0(\mathcal{O}(2)^2)$ has degree ≤ 1. If now (F, W) is a coherent subsystem of (E, V) with $\mathrm{rk} F = 1$, we have either $\dim W = 0$, $\deg F = 2$ or $\dim W \leq 1$, $\deg F \leq 1$. In the first case, the α-stability condition holds for $\alpha > \frac{1}{3}$, in the second case for all $\alpha > 0$. Now suppose (F, W) is a coherent subsystem of (E, V) with $\mathrm{rk} F = 2$. Then either $\dim W \leq 1$, $\deg F = 4$ or $\dim W \leq 2$, $\deg F \leq 3$. In the first case, the α-stability condition holds for $\alpha > \frac{2}{3}$, in the second case for all $\alpha > 0$. Thus we have shown that (E, V) is α-stable for $\alpha > \frac{2}{3}$.

Conversely suppose (E, V) is an α-stable coherent system of type $(3, 5, 3)$. Then $E \simeq \mathcal{O}(2)^2 \oplus \mathcal{O}(1)$ or $\mathcal{O}(3) \oplus \mathcal{O}(1)^2$. Since $h^0(\mathcal{O}(1)) = 2$, (E, V) has a coherent subsystem of type $(2, 4, 1)$, which contradicts α-stability for $\alpha \leq \frac{2}{3}$. □

10. An example for $k = 4$

In this section we give an example of an allowable critical data set A_c with $0 < k < n$ and $C_{12} = 0$. According to our earlier results, we

must have $k \geq 4$ and, by Proposition 5.3, $k_1 < n_1$. Now (4) implies that $k_2 < n_2$, so $n \geq 6$. The minimal possible example therefore has $n = 6$, $k = 4$ and one can check that then $k_1 = 3$, $k_2 = 1$, $n_1 = 4$, $n_2 = 2$. The formula (32) gives

$$kC_{12} = (n_1 - k_1)(nk_2 l + k_2 t - kn_2) + (n - k_1)f + k_1 n_2 m - kk_1 k_2,$$

i. e.

$$4C_{12} = 6l + t + 3f + 6m - 20.$$

Now $l \geq 1$ and, by (13), $f \equiv m \bmod 4$. Since $f \geq 1$, either f and m are both positive or $f \geq 4$. It is now easy to check that the only cases giving $C_{12} = 0$ are

$$l = 1, f = m = 1, t = 5; \quad l = 1, f = 4, m = 0, t = 2.$$

In both cases one can check from (12) that $e = -1$, which implies by (11) that $t \geq 5$. Thus we are left with just one case in which a simple computation gives $d = 7$. Note in this case that the necessary condition for α-stability from [2, Propositions 4.1 and 4.2] is

$$\frac{5}{4} < \alpha < \frac{11}{4}.$$

Now by (9)

$$\alpha_c = \frac{ne + n_2 t}{n_2 k - nk_2} = \frac{-6 + 10}{8 - 6} = 2,$$

which does lie within the given range. The critical data set itself is given by

$$A_c = (\alpha_c, n_1, d_1, k_1, n_2, d_2, k_2) = (2, 4, 4, 3, 2, 3, 1).$$

One can check (30) and (31) to show that A_c is allowable.

Proposition 10.1. (a): $G(\alpha; 6, 7, 4) \neq \emptyset$ *for* $\frac{5}{4} < \alpha < 2$.
(b): $G(\alpha; 6, 7, 4) = \emptyset$ *for* $\alpha \geq 2$.

Proof. We show first that $G(2^+; 6, 7, 4) = \emptyset$ and $G(2^-; 6, 7, 4) \neq \emptyset$. First note that

$$C_{21} = -n_1 n_2 + d_2 n_1 - d_1 n_2 + k_2(d_1 + n_1 - k_1) = -8 + 12 - 8 + 4 + 4 - 3 = 1.$$

The result will therefore follow from Corollary 3.5 and Remarks 3.7 and 3.8 if we prove the existence of 2-stable coherent systems of types $(4, 4, 3)$ and $(2, 3, 1)$. In the first case, we have to check the conditions of Theorem 8.4; in the second case, those of [2, Theorem 5.1]. Both computations are easy.

By Corollary 3.5, we have $G(2^-; 6, 7, 4) = G_2^-$, so $G(2; 6, 7, 4) = \emptyset$. It follows from [2, Corollary 3.4] that $G(\alpha; 6, 7, 4) = \emptyset$ if $\alpha \geq 2$, thus proving (b). For (a), we can apply Proposition 5.12 to show that $G((5/4)^+; 6, 7, 4) \neq \emptyset$. In particular, we must show that $C_{12} > 0$ for all allowable critical data sets for coherent systems of type $(5, 5, 4)$. From our argument above, we have shown that $C_{12} > 0$ always if $k = 4$

and $n = 5$, so this is clear. The result now follows from [2, Corollary 3.4]. □

References

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